Advanced Math

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6) $3+7+11+15+\cdots+(4n-1) = n(2n+1)$

1) $S_1 = 1(2(1)+1) = 3$ ok

2) Assume $S_k = k(2k+1)$ is true, then $S_{k+1} = (k+1)(2(k+1)+1) = (k+1)(2k+3)$

3) Show $S_k + a_{k+1} = S_{k+1}$; $k(2k+1) + [4(k+1)-1]$

$= 2k^2+k+4k+3$

$= 2k^2+5k+3 = (k+1)(2k+3) = S_{k+1}$ □

8) $1+4+7+10+\cdots+(3n-2) = \frac{n}{2}(3n-1)$

1) $S_1 = \frac{1}{2}(3(1)-1) = 1$ ok

2) Assume $S_k = \frac{k}{2}(3k-1)$ is true, then $S_{k+1} = \frac{k+1}{2}(3(k+1)-1) = \frac{(k+1)(3k+2)}{2}$

3) $S_k + a_{k+1} = \frac{k}{2}(3k-1) + 3(k+1)-2$

$= \frac{3k^2-k}{2} + \frac{2(3k+1)}{2} = \frac{3k^2-k+6k+2}{2} = \frac{(3k^2+5k+2)}{2} = \frac{(3k+2)(k+1)}{2} = S_{k+1}$ □

10) $2(1+3+3^2+3^3+\cdots+3^{n-1}) = 3^n-1$

1) $S_1 = 3^{(1)}-1 = 2 = 2(1)$ ok

2) Assume $S_k = 3^k-1$ is true, then $S_{k+1} = 3^{k+1}-1$

3) $S_k + a_{k+1} = 3^k-1 + 2(3^{k+1-1})$

$= 3^k-1+2\cdot3^k = 3\cdot3^k-1 = 3^1\cdot3^k-1 = 3^{k+1}-1 = S_{k+1}$ □

12 $1^2+2^2+3^2+4^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}$

1) $S_1 = \frac{1(1+1)(2(1)+1)}{6} = \frac{1(2)(3)}{6} = \frac{6}{6} = 1 = 1^2$ ok

2) Assume $S_k = \frac{k(k+1)(2k+1)}{6}$ is true, then $S_{k+1} = \frac{(k+1)(k+1+1)[2(k+1)+1]}{6} = \frac{(k+1)(k+2)(2k+3)}{6}$

3) $S_k + a_{k+1} = \frac{k(k+1)(2k+1)}{6} + (k+1)^2$

$= \frac{k(k+1)(2k+1)}{6} + \frac{6(k+1)^2}{6} = \frac{k(k+1)(2k+1)+6(k+1)^2}{6}$

$= \frac{(k+1)[k(2k+1)+6(k+1)]}{6}$

$= \frac{(k+1)(2k^2+k+6k+6)}{6} = \frac{(k+1)(2k^2+7k+6)}{6} = \frac{(k+1)(2k+3)(k+2)}{6} = S_{k+1}$ □

14) $(1+\frac{1}{1})(1+\frac{1}{2})(1+\frac{1}{3})\cdots(1+\frac{1}{n}) = n+1$

1) $S_1 = 1+1 = 1+\frac{1}{1}$ ok

2) Assume $S_k = k+1$ is true, then $S_{k+1} = (k+1)+1 = k+2$

3) $(S_k)(a_{k+1}) = (k+1)(1+\frac{1}{k+1})$

$= k+1 + 1 = k+2 = S_{k+1}$ □

Hint

$(k+1)(1+\frac{1}{k+1})$

distribute the whole thing! Don't FOIL

16) $\sum_{i=1}^{n} i^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$

1) $S_1 = \frac{1(1+1)[2(1)+1][3(1)^2+3(1)-1]}{30} = \frac{1(2)(3)(5)}{30} = 1 = 1^4$ ok

2) If $S_k = \frac{k(k+1)(2k+1)(3k^2+3k-1)}{30}$, then $S_{k+1} = \frac{(k+1)(k+2)(2k+3)(3(k+1)^2+3(k+1)-1)}{30}$

$3(k^2+2k+1)$
$3k^2+6k+3$
$+3k+3-1$

$= \frac{(k+1)(k+2)(2k+3)(3k^2+9k+5)}{30}$

3) $S_k + a_{k+1} = \frac{k(k+1)(2k+1)(3k^2+3k-1)}{30} + \frac{30(k+1)^4}{30} = (k+1)\left[\frac{(2k^2+k)(3k^2+3k-1)+30(k^3+3k^2+3k+1)}{30}\right]$

$= (k+1)\left[\frac{6k^4+6k^3-2k^2+3k^3+3k^2-k+30k^3+90k^2+90k+30}{30}\right] = (k+1)\left[\frac{6k^4+39k^3+91k^2+89k+30}{30}\right]$

	6	39	91	89	30	
		-12	-54	-74	-30	
-2	6	27	37	15	0	⇐ $(k+2)$ is a Factor
		-9	-27	-15		
$-\frac{3}{2}$	6	18	10	0		⇐ $(2k+3)$ is a Factor

$6x^2+18x+10=0 \rightarrow 3x^2+9x+10=0 \Rightarrow (3x^2+9x+10)$ is a Factor

$\Rightarrow = \frac{(k+1)(k+2)(2k+3)(3x^2+9x+10)}{30} = S_{k+1}$ □

18) $\sum_{i=1}^{n} \frac{1}{(2i+1)(2i-1)} = \frac{n}{2n+1}$

1) $S_1 = \frac{1}{2(1)+1} = \frac{1}{3} = \frac{1}{(2+1)(2-1)}$ ok

2) If $S_k = \frac{k}{2k+1}$, then $S_{k+1} = \frac{k+1}{2(k+1)+1} = \frac{k+1}{2k+3}$

3) $S_k + a_{k+1} = \frac{k}{2k+1} + \frac{1}{[2(k+1)+1][2(k+1)-1]} = \frac{k}{2k+1} + \frac{1}{(2k+3)(2k+1)} = \frac{k(2k+3)+1}{(2k+3)(2k+1)}$

$= \frac{2k^2+3k+1}{(2k+3)(2k+1)} = \frac{(2k+1)(x+1)}{(2k+3)(2k+1)} = \frac{x+1}{2k+3} = S_{k+1}$ □

36) $\left(\frac{4}{3}\right)^n > n$, $n \geq 7$ 1) True for $n=7$? $\left(\frac{4}{3}\right)^7 > 7 \Rightarrow 7.492 > 7$ ok

2) IF $\left(\frac{4}{3}\right)^k > k$, then $\left(\frac{4}{3}\right)^{k+1} > k+1$

3) $\left(\frac{4}{3}\right)^k > k \Rightarrow \left(\frac{4}{3}\right)^k\left(\frac{4}{3}\right) > k\left(\frac{4}{3}\right)$

$\Rightarrow \left(\frac{4}{3}\right)^{k+1} > \left(\frac{4}{3}\right)k$

But since $n \geq 7$ $\frac{4}{3}(7) = \frac{28}{3} = 9\frac{1}{3}$ and $7+1=8$ (diff $1\frac{1}{3}$)

$\frac{4}{3}(8) = \frac{32}{3} = 10\frac{2}{3}$ and $8+1=9$ (diff $1\frac{2}{3}$)

$\frac{4}{3}(9) = \frac{36}{3} = 12$ and $9+1=10$ (diff by 2)

but $\frac{4}{3}k = k + \frac{1}{3}k \geq k + \frac{7}{3}$ (since $k>7$) $\Rightarrow k+\frac{7}{3} > k+1$ ∴

$\Rightarrow \frac{4}{3}k > k+1$

$\Rightarrow \left(\frac{4}{3}\right)^{k+1} > k+1$ □

38) $\left(\frac{x}{y}\right)^{n+1} < \left(\frac{x}{y}\right)^n$ if $n \geq 1$ and $0<x<y$

Note: Since $0<x<y$, $\frac{x}{y}<1$ and $\frac{y}{x}>1$

1) True for $n=1$? $\left(\frac{x}{y}\right)^{1+1} < \left(\frac{x}{y}\right)^1 \Rightarrow \left(\frac{x}{y}\right)^2 < \left(\frac{x}{y}\right)$

From Given $\left(\frac{x}{y}\right)<1 \Rightarrow \left(\frac{x}{y}\right)\left(\frac{x}{y}\right) < 1\left(\frac{x}{y}\right)$ ∵ $x>0$, $y>0$ so no sign switch ∴

$= \left(\frac{x}{y}\right)^2 < \frac{x}{y}$ OK

2) IF $\left(\frac{x}{y}\right)^{k+1} < \left(\frac{x}{y}\right)^k$, then $\left(\frac{x}{y}\right)^{k+2} < \left(\frac{x}{y}\right)^{k+1}$

3) $\left(\frac{x}{y}\right)^{k+1} < \left(\frac{x}{y}\right)^k \Rightarrow \left(\frac{x}{y}\right)^{k+1}\left(\frac{x}{y}\right) < \left(\frac{x}{y}\right)^k\left(\frac{x}{y}\right)$, $x>0$, $y>0$, no sign switch

$\Rightarrow \left(\frac{x}{y}\right)^{k+2} < \left(\frac{x}{y}\right)^{k+1}$

45) $\sin(x+n\pi) = (-1)^n \sin x$

1) True for $n=1$? $\sin(x+1\pi) = (-1)^1 \sin x$

$\sin x \cos\pi + \sin\pi \cos x = -\sin x$

$-\sin x + 0\cos x = -\sin x$ OK

2) IF $\sin(x+k\pi) = (-1)^k \sin x$, then $\sin(x+(k+1)\pi) = (-1)^{k+1}\sin x$

LOOK

3) $\sin(x+k\pi) = (-1)^k \sin x$

$\sin[(x+k\pi)+\pi] = \sin(x+k\pi)\cos\pi + \sin\pi\cos(x+k\pi) = \left[(-1)^k \sin x\right](-1) + 0\cos(x+k\pi)$

$= (-1)(-1)^k \sin x = (-1)^{k+1}\sin x$ □

46) $\tan(x+n\pi) = \tan x$

1) True for $n=1$? $\tan(x+1\pi) = \frac{\tan x + \tan\pi}{1-\tan x\tan\pi} = \frac{\tan x + 0}{1-0} = \tan x$ ok

2) If $\tan(x+k\pi) = \tan x$, then $\tan(x+(k+1)\pi) = \tan x$

3) $\tan(x+k\pi) = \tan x$, therefore

$$\tan[(x+k\pi)+\pi] = \frac{\tan(x+k\pi)+\tan\pi}{1+\tan(x+k\pi)\tan\pi} = \frac{\tan(x+k\pi)+0}{1+\tan(x+k\pi)0}$$

$$= \frac{\tan(x+k\pi)}{1} = \frac{\tan x}{1} \quad \square$$